

A VARIATIONAL FORMULATION FOR MODELING A
BOSE-EINSTEIN CONDENSATION IN AN
EULER-BERNOULLIAN CONTEXT

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Abstract

This article develops a variational formulation for modeling a Bose-Einstein condensation process. The results are based on standard tools of calculus of variations and optimization theory. Moreover, we emphasize the context here addressed is essentially an Euler-Bernoullian one.

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1 Introduction

The first part of this article develops a variational formulation for modeling a Bose-Einstein condensation in an Euler-Bernoullian context.

Such results are based on tools of calculus of variations and related Lagrange multipliers approach. It is worth emphasizing, in the Bose-Einstein condensation model here presented, a particle system corresponding to a fluid phase a , as submitted to a very low temperature (close to absolute zero), partially condensate into a phase b , known as the Bose-Einstein condensate. For such a condensed phase b , the particle system presents a single eigenvalue energy level, which depends only on the time t (through a temperature dependence) but does not depend on the macroscopic spatial variable y . In summary, the condensed phase b behaves somewhat like a "single particle field", with a unique quantum energy level.

Hence, concerning such scenarios, the particle mass constraint functionals for the phases a e b stand for

$$J_{Aux_1^a} = \int_0^{t_f} \int_{\Omega} E_1^a(y, t) \left(\int_{\Omega} (|\phi_a(x, y, t)|^2 dx - m_p^a) \right) dy dt, \quad (1)$$

where for such a phase a , the energy level $E_1^a(y, t)$ depends on y and t , and

$$J_{Aux_1^b} = \int_0^{t_f} E_1^b(t) \left(\int_{\Omega} (|\phi_b(x, t)|^2 dx - m_p^b) \right) dt, \quad (2)$$

where for such a phase b , the energy level $E_1^b(t)$ does not depend on y , that is, depends only on t , qualitatively behaving as a single energy state at each t .

Furthermore, we highlight the two main fields for such a system are a position one and a velocity one, which we specify below in the next lines.

Similar and related models in physics are addressed in [1, 2].

As standard references in theoretical fluid mechanics we would cite [3, 4, 5, 6, 7]. For related numerical methods we would mention [8, 10, 11]. Concerning other similar models, we would cite [12, 13, 14, 15, 16, 17].

Finally, details on the Sobolev spaces involved may be found in [18].

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and simply connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Generically, we consider a fluid motion initially modeled as a particle system one, through the field of position $\mathbf{r} : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3$ and field of velocities $\mathbf{v}$ where $\mathbf{v}(\mathbf{r}(x, t), t)$ stands for the velocity at the position $\mathbf{r}(x, t)$ and time $t \in [0, t_f]$.

Moreover, the gravity field is denoted by

$$\mathbf{g} = -g\mathbf{k},$$

related to the $\mathbb{R}^3$ basis

$$\{\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)\}.$$

2 The mathematical model

In this section, we develop in details the concerning variational formulation for a compressible case.

We emphasize such a set $\Omega \subset \mathbb{R}^3$ stands for a control volume, in which along a time interval $[0, t_f]$, where $t \in [0, t_f]$ denotes time, a fluid particle system develops a compressible motion.

For a particle system corresponding to a phase a , we start by defining the following density field,

$$\hat{\rho}_a(x, y, t) = \left(|\phi_a(x, y, t)|^2 \frac{|\phi_a^M(y, t)|^2}{m_p^a} \right) \quad (3)$$

Here m_p^a denotes the mass of a single particle type a .

The macroscopic fluid density for such phase a , is defined by

$$\rho_a(y, t) = |\phi_a^M(y, t)|^2.$$

We recall that in the Bose-Einstein condensation, the condensed phase b has a unique energy level, which depends only on t but does not depend on the macroscopic variable y .

For such a particle system corresponding to the condensed phase b , we start by defining the following density field,

$$\hat{\rho}_b(x, y, t) = \left(|\phi_b(x, t)|^2 \frac{|\phi_b^M(y, t)|^2}{m_p^b} \right) \quad (4)$$

Here m_p^b denotes the mass of a single particle type b .

The macroscopic fluid density for such phase b , is defined by

$$\rho_b(y, t) = |\phi_b^M(y, t)|^2.$$

It is worth highlighting that generically, we denote

$$L^2(\Omega \times [0, t_f]) = \left\{ h : \Omega \times [0, t_f] \rightarrow \mathbb{R} \text{ measurable} : \int_0^{t_f} \int_{\Omega} |h(x, t)|^2 dx dt < +\infty \right\},$$

$$\begin{aligned} L^2(\Omega \times [0, t_f]; \mathbb{R}^3) &= \{ h = (h_1, h_2, h_3) : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3 \\ &: h_k \in L^2(\Omega \times [0, t_f]), \forall k \in \{1, 2, 3\} \}, \end{aligned} \quad (5)$$

$$\begin{aligned} W^{1,2}(\Omega \times [0, t_f]) &= \{ h : \Omega \times [0, t_f] \rightarrow \mathbb{R} : h \in L^2(\Omega \times [0, t_f]) \\ &\text{and } h_t, h_{x_j} \in L^2(\Omega \times [0, t_f]), \forall j \in \{1, 2, 3\} \}, \end{aligned} \quad (6)$$

where the partial derivatives h_t, h_{x_j} are understood in a distributional sense, $\forall j \in \{1, 2, 3\}$.

Moreover,

$$\begin{aligned} W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) &= \{ h = (h_1, h_2, h_3) : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3 \\ &: h_k \in W^{1,2}(\Omega \times [0, t_f]), \forall k \in \{1, 2, 3\} \} \end{aligned} \quad (7)$$

Similar notations are valid for other sets and other related complex and vectorial cases.

With such notations in mind, we define the concerning function spaces

$$\begin{aligned} V_1^a &= \\ &\left\{ \phi_a \in W^{1,2}(\Omega \times \Omega \times [0, t_f]; \mathbb{C}) : \nabla_x \phi_a \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times \Omega \times [0, t_f], \right. \\ &\nabla_y \phi_a \cdot \mathbf{n} = 0 \text{ on } \Omega \times \partial\Omega \times [0, t_f], \\ &\left. \frac{\partial \phi_a(x, y, 0)}{\partial t} = 0 \text{ and } \frac{\partial \phi_a(x, y, t_f)}{\partial t} = 0 \text{ in } \Omega \times \Omega \right\} \end{aligned} \quad (8)$$

and

$$\begin{aligned} V_2^a &= \left\{ \phi_a^M \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \phi_a^M = 0 \text{ on } \partial(\Omega_1 \cup \Omega_2) \times [0, t_f], \right. \\ &\left. \phi_a^M(z, t) = \phi_{in}(z, t) \text{ on } \partial\Omega_{in} \times [0, t_f] \text{ and } \phi_a^M(z, 0) = \phi_0 \text{ in } \Omega \right\}, \end{aligned} \quad (9)$$

where

$$\partial\Omega = \partial\Omega_{in} \cup \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_{out}.$$

Moreover we suppose such a union is quasi-disjoint and through $\partial\Omega_{in}$ and $\partial\Omega_{out}$ are allowed the entering and exiting of fluid mass, respectively.

We define also,

$$V_1^b = \left\{ \phi_b \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \nabla_x \phi_b \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times [0, t_f], \right. \\ \left. \frac{\partial \phi_b(x, 0)}{\partial t} = 0 \text{ and } \frac{\partial \phi_b(x, t_f)}{\partial t} = 0 \text{ in } \Omega \right\} \quad (10)$$

and

$$V_2^b = \left\{ \phi_b^M \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \phi_b^M = 0 \text{ on } \partial(\Omega_1 \cup \Omega_2) \times [0, t_f], \right. \\ \left. \phi_b^M(z, t) = \phi_{in}(z, t) \text{ on } \partial\Omega_{in} \times [0, t_f] \text{ and } \phi_b^M(z, 0) = \phi_0 \text{ in } \Omega \right\}, \quad (11)$$

Observe that, we may denote

$$\rho(y, t) = \rho_a(y, t) + \rho_b(y, t).$$

2.1 The position fields

For the phase a , we start by denoting the macroscopic electronic position field as

$$\mathbf{r}_a : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3,$$

where

$$\mathbf{r}_a(x, t) = (X_a(x, t), Y_a(x, t), Z_a(x, t)).$$

In a internal variable and relativistic context, the particle position field for such a phase a , is defined by

$$\hat{\mathbf{r}}_a(x, t) = (\mathbf{r}_a(x, t), t) + \tilde{\mathbf{r}}_a^p(\mathbf{r}_a(x, t), t) = (\mathbf{r}_a(x, t), t) + (ct, \tilde{\mathbf{r}}_a(\mathbf{r}_a(x, t))).$$

Here $c > 0$ denotes the speed of light.

For the phase b , we start by denoting the macroscopic electronic position field as

$$\mathbf{r}_b : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3,$$

where

$$\mathbf{r}_b(x, t) = (X_b(x, t), Y_b(x, t), Z_b(x, t)).$$

In a internal variable and relativistic context, the particle position field for such a phase a , is defined by

$$\hat{\mathbf{r}}_b(x, t) = (\mathbf{r}_b(x, t), t) + \tilde{\mathbf{r}}_b^p(\mathbf{r}_b(x, t), t) = (\mathbf{r}_b(x, t), t) + (ct, \tilde{\mathbf{r}}_b(\mathbf{r}_b(x, t))).$$

We define also the following function space

$$\begin{aligned} V_3^a &= \{\mathbf{r}_a \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) : \mathbf{r}_a = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, t_f] \\ &\quad \text{and } \mathbf{r}_a(z, 0) = \mathbf{r}_0 \text{ in } \Omega\}, \end{aligned} \quad (12)$$

and assume

$$\tilde{\mathbf{r}}_a \in V_4 = W^{1,2}(\mathbb{R}^3; \mathbb{R}^3).$$

For the phase b , we define the following corresponding space

$$\begin{aligned} V_3^b &= \{\mathbf{r}_b \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) : \mathbf{r}_b = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, t_f] \\ &\quad \text{and } \mathbf{r}_b(z, 0) = \mathbf{r}_0 \text{ in } \Omega\}, \end{aligned} \quad (13)$$

and assume

$$\tilde{\mathbf{r}}_b \in V_4 = W^{1,2}(\mathbb{R}^3; \mathbb{R}^3).$$

2.2 About the system constraints

Denoting by $m_T = m_a(0) + m_b(0)$ the total system mass at $t = 0$, we have the following constraint

$$\begin{aligned} m_T &= m_a(0) + m_b(0) \\ &= m_a(t) + \int_0^t \int_{\partial\Omega} \rho_a(y, \tau) \mathbf{v}_a \cdot \mathbf{n} \, dS \, d\tau \\ &\quad + m_b(t) + \int_0^t \int_{\partial\Omega} \rho_b(y, \tau) \mathbf{v}_b \cdot \mathbf{n} \, dS \, d\tau \end{aligned} \quad (14)$$

where

$$\begin{aligned} \mathbf{v}_a &= \mathbf{v}_a(\mathbf{r}_a(x, t), t), \\ \mathbf{v}_b &= \mathbf{v}_b(\mathbf{r}_b(x, t), t). \end{aligned}$$

and where

$$m_a(t) = \int_{\Omega} \rho_a(y, t) \, dy$$

and

$$m_b(t) = \int_{\Omega} \rho_b(y, t) \, dy,$$

on $[0, t_f]$.

Here $\mathbf{n}$ denotes outward normal field to $\partial\Omega = S$.

Relating the internal kinetics energy, in a relativistic context, we define the following functional

$$\begin{aligned}
& E_{C_1}^a \\
&= -\frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \frac{\hat{\rho}_a(x, y, t) d\tilde{\mathbf{r}}_a}{\sqrt{1 - \frac{\tilde{v}_a^2}{c^2}}} \cdot \frac{d\tilde{\mathbf{r}}_a}{dt} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \frac{\hat{\rho}_a(x, y, t)}{\sqrt{1 - \frac{\tilde{v}_a^2}{c^2}}} (c^2 - \tilde{v}_a^2) \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - \tilde{v}_a^2} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - g_{jk}^a \frac{\partial(X_a)_j}{\partial t} \frac{\partial(X_a)_k}{\partial t}} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} e_C^a dy dt, \tag{15}
\end{aligned}$$

where generically, we have denoted, $X_0 = t$,

$$g_{jk}^a = \frac{\partial \tilde{\mathbf{r}}_a}{\partial (X_a)_j} \cdot \frac{\partial \tilde{\mathbf{r}}_a}{\partial (X_a)_k}$$

$g^a = \det\{g_{jk}^a\}$, where

$$w \cdot v = -w_0 v_0 + \sum_{k=1}^3 w_k v_k$$

for

$$w = (w_0, w_1, w_2, w_3)$$

and

$$v = (v_0, v_1, v_2, v_3) \in \mathbb{R}^4,$$

and

$$e_C^a = \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - g_{jk}^a \frac{\partial(X_a)_j}{\partial t} \frac{\partial(X_a)_k}{\partial t}} \sqrt{-g^a} |\det X'_a| dx.$$

With analogous notation, the corresponding functional for the phase b is defined by

$$\begin{aligned}
& E_{C_1}^b \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_b(x, y, t) \sqrt{c^2 - g_{jk}^b \frac{\partial(X_b)_j}{\partial t} \frac{\partial(X_b)_k}{\partial t}} \sqrt{-g^b} |\det X'_b| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} e_C^b dy dt, \tag{16}
\end{aligned}$$

where,

$$e_C^b = \int_{\Omega} c \hat{\rho}_b(x, y, t) \sqrt{c^2 - g_{jk}^b \frac{\partial(X_b)_j}{\partial t} \frac{\partial(X_b)_k}{\partial t}} \sqrt{-g^b} |\det X'_b| dx.$$

For the interacting and self-interacting energies, in a non-linear Schrödinger equation context, we define the following functional,

$$\begin{aligned} E_{in} = & \frac{\alpha_1^a}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_a(y, t)^2 dy dt \\ & - \frac{\beta_1^a}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_a(y, t) dy dt \\ & + \frac{\alpha_1^b}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_b(y, t)^2 dy dt \\ & - \frac{\beta_1^b}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_b(y, t) dy dt \end{aligned} \quad (17)$$

for appropriate $\alpha_1^a > 0$, α_1^b and $\beta_1^a, \beta_1^b > 0$.

On the other hand, we define

$$\begin{aligned} E_M = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \left(\frac{\gamma_1^a}{m_p^a} |\nabla \phi_a(x, y, t)|^2 \right) |\phi_a^M(y, t)|^2 dx dy dt \\ = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \left(\frac{\gamma_1^b}{m_p^b} |\nabla \phi_b(x, t)|^2 \right) |\phi_b^M(y, t)|^2 dx dy dt, \end{aligned} \quad (18)$$

for appropriate constants $\gamma_1^a, \gamma_1^b > 0$.

Concerning the restrictions, already including the respective Lagrange multipliers, we define also the following functionals,

$$\begin{aligned} J_{Aux_1^a} = & \int_0^{t_f} \int_{\Omega} E_1^a(y, t) \left(\int_{\Omega} (|\phi_a(x, y, t)|^2 dx - m_p^a) \right) dy dt, \end{aligned} \quad (19)$$

$$\begin{aligned} J_{Aux_1^b} = & \int_0^{t_f} E_1^b(t) \left(\int_{\Omega} (|\phi_b(x, t)|^2 dx - m_p^b) \right) dt, \end{aligned} \quad (20)$$

$$J_{Aux_2} = \int_0^{t_f} E_2(t) \left(m_T - \int_{\Omega} \rho_a(y, t) dy - \int_0^t \int_{\partial\Omega} \rho_a(y, \tau) \mathbf{v}_a \cdot \mathbf{n} dS d\tau \right. \\ \left. - \int_{\Omega} \rho_b(y, t) dy - \int_0^t \int_{\partial\Omega} \rho_b(y, \tau) \mathbf{v}_b \cdot \mathbf{n} dS d\tau \right) dt. \quad (21)$$

$$J_{Aux_3^a} = \int_0^{t_f} \int_{\Omega} E_3^a(x, t) \left(\mathbf{v}_a(\mathbf{r}_a(x, t), t) - \frac{\partial \mathbf{r}_a(x, t)}{\partial t} \right) dx dt, \\ J_{Aux_3^b} = \int_0^{t_f} \int_{\Omega} E_3^b(x, t) \left(\mathbf{v}_b(\mathbf{r}_b(x, t), t) - \frac{\partial \mathbf{r}_b(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_4} = - \int_0^{t_f} \int_{\Omega} \hat{P} \left(\frac{d\rho}{dt} + \rho_a \operatorname{div} \mathbf{v}_a + \rho_b \operatorname{div} \mathbf{v}_b \right) dy dt, \quad (22)$$

Moreover, considering the pressure and temperature scalar fields, we define

$$P_a = \frac{d(\hat{P}\rho_a)}{dt}, \\ P_b = \frac{d(\hat{P}\rho_b)}{dt},$$

where

$$\hat{P} \in V_5$$

$$V_5 = \{ \hat{P} \in W^{1,2}(\Omega \times [0, t_f]) : P = P_a + P_b = P(\hat{P}, \rho) = P_0, \text{ on } \partial\Omega \times [0, t_f] \}.$$

and the temperature fields, T_a, T_b through the equations

$$(C_v)_a \rho_a T_a = e_C^a \quad (23)$$

so that

$$(C_v)_b \rho_b T_b = e_C^b \quad (24)$$

for an appropriate real constant $C_v > 0$.

At this point, we define the entropy differentials

$$dS_a = -\hat{K}_a \frac{\rho_a}{\rho} \ln \left(\frac{\rho_a}{e \rho} \right) dy dt,$$

$$dS_b = -\hat{K}_b \frac{\rho_b}{\rho} \ln \left(\frac{\rho_b}{e \rho} \right) dy dt,$$

and the functional

$$E^S = \int_0^{t_f} \int_{\Omega} (T_a dS_a + T_b dS_b).$$

Moreover, we also define

$$e^S = T_a S_a + T_b S_b,$$

where

$$S_a = -\hat{K}_a \frac{\rho_a}{\rho} \ln \left(\frac{\rho_a}{e \rho} \right)$$

and

$$S_b = -\hat{K}_b \frac{\rho_b}{\rho} \ln \left(\frac{\rho_b}{e \rho} \right).$$

With such results in mind, we define the restriction

$$\begin{aligned} & (C_v)_a \rho_a \frac{dT_a}{dt} + (C_v)_b \rho_b \frac{dT_b}{dt} \\ &= K_a \nabla^2 T_a + K_b \nabla^2 T_b \\ & \quad - P_a \operatorname{div} \mathbf{v}_a - P_b \operatorname{div} \mathbf{v}_b + \frac{de^S}{dt}, \end{aligned} \tag{25}$$

for an appropriate real constant $K_a > 0$, $K_b > 0$.

At this point, we define the following function space

$$\begin{aligned} V_6^a &= \{T_a \in W^{1,2}(\Omega \times [0, t_f]) : \\ & T_a(z, t) = T_1^a(z, t) \text{ on } \partial\Omega \times [0, t_f] \text{ and } T_a(z, 0) = T_0^a(z) \text{ in } \Omega\}, \end{aligned} \tag{26}$$

$$\begin{aligned} V_6^b &= \{T_b \in W^{1,2}(\Omega \times [0, t_f]) : \\ & T_b(z, t) = T_1^b(z, t) \text{ on } \partial\Omega \times [0, t_f] \text{ and } T_b(z, 0) = T_0^b(z) \text{ in } \Omega\}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} V_7^a &= \{\mathbf{v}_a \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) : \mathbf{v}_a = \mathbf{0} \text{ on } (\partial\Omega_1 \cup \partial\Omega_2) \times [0, t_f] \\ & \text{and } \mathbf{v}_a(z, 0) = \mathbf{v}_0^a \text{ on } \Omega\}. \end{aligned} \tag{28}$$

$$\begin{aligned} V_7^b &= \{\mathbf{v}_b \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) : \mathbf{v}_b = \mathbf{0} \text{ on } (\partial\Omega_1 \cup \partial\Omega_2) \times [0, t_f] \\ & \text{and } \mathbf{v}_b(z, 0) = \mathbf{v}_0^b \text{ on } \Omega\}. \end{aligned} \tag{29}$$

With such restrictions and definitions in mind, we define also the following functionals,

$$J_{Aux_5^a} = \int_0^{t_f} \int_{\Omega} E_5^a(y, t) ((C_v)_a \rho_a T_a - e_C^a) dy dt \quad (30)$$

$$J_{Aux_5^b} = \int_0^{t_f} \int_{\Omega} E_5^b(y, t) ((C_v)_b \rho_b T_b - e_C^b) dy dt \quad (31)$$

and

$$\begin{aligned} J_{Aux_6} = & \int_0^{t_f} \int_{\Omega} E_6(y, t) \left((C_v)_a \rho_a \frac{dT_a}{dt} + (C_v)_b \rho_b \frac{dT_b}{dt} \right. \\ & - K_a \nabla^2 T_a - K_b \nabla^2 T_b \\ & \left. + P_a \operatorname{div} \mathbf{v}_a + P_b \operatorname{div} \mathbf{v}_b - \frac{de^S}{dt} \right) dy dt. \end{aligned} \quad (32)$$

Finally, we define the following functional,

$$\begin{aligned} J_{Aux_7^a} = & \int_0^{t_f} \int_{\Omega} E_{50}^a(y, t) \\ & \times \left(\rho_a(y, t) - \int_{\Omega} \left(\frac{|\phi_a(x, y, t)|^2}{m_p^a} \right) dx |\phi_a^M(y, t)|^2 \right) dy dt. \end{aligned} \quad (33)$$

$$\begin{aligned} J_{Aux_7^b} = & \int_0^{t_f} \int_{\Omega} E_{50}^b(y, t) \\ & \times \left(\rho_b(y, t) - \int_{\Omega} \left(\frac{|\phi_b(x, y, t)|^2}{m_p^b} \right) dx |\phi_b^M(y, t)|^2 \right) dy dt. \end{aligned} \quad (34)$$

3 The main functional

Here we define the functional J , by

$$\begin{aligned} J = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \rho_a \mathbf{v}_a \cdot \mathbf{v}_a dy dt - \int_0^{t_f} \int_{\Omega} \rho_a \mathbf{g} \cdot \mathbf{r}_a dy dt \\ & + \frac{1}{2} \int_0^{t_f} \int_{\Omega} \rho_b \mathbf{v}_b \cdot \mathbf{v}_b dy dt - \int_0^{t_f} \int_{\Omega} \rho_b \mathbf{g} \cdot \mathbf{r}_b dy dt. \end{aligned} \quad (35)$$

Denoting

$$V = V_1^a \times V_1^b \times V_2^a \times V_2^b \times V_3^a \times V_3^b \times V_5 \times V_6^a \times V_6^b \times V_7^a \times V_7^b,$$

the final variational formulation $J_{total} : V \rightarrow \mathbb{R}$ stands for

$$\begin{aligned}
 J_{total} = & -J + E_{C_1}^a + E_{C_1}^b + E_{in} \\
 & + E_M + J_{Aux_1^a} + J_{Aux_1^b} + J_{Aux_2} + J_{Aux_3^a} + J_{Aux_3^b} \\
 & J_{Aux_4} + J_{Aux_5^a} + J_{Aux_5^b} \\
 & + J_{Aux_6} + J_{Aux_7^a} + J_{Aux_7^b} - E^S.
 \end{aligned} \tag{36}$$

The objective of this section is complete.

4 Conclusion

In this article, we have presented a functional which the variation leads to a system of partial differential equations intended to model a Bose-Einstein condensation chemical reaction.

In a near future research we intend to address similar models in superfluidity.

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